Linear Transformations

Finally, we get to our last, absolute last, topic. Yay!

Two components: 'Transformation' and 'Linear'. We'll look at them in order.

Definition: A Transformation $T : V \to W$, links every $\mathbf{x}$ in vector space V to $T(\mathbf{x})$ in vector space W. It's only a transformation if each vector $\mathbf{x} \in V$ gets linked to a SINGLE vector in W, called the image of $\mathbf{x}$.

Definition: A Transformation $T : V \to W$ is a LINEAR Transformation if

- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$.
- $T(a\mathbf{x}) = aT(\mathbf{x})$ for all $\mathbf{x} \in V$ and $a \in \mathbb{R}$.

We will primarily look into transformations $T : \mathbb{R}^n \to \mathbb{R}^m$.

Note: 'Linear Transformation' may SOUND like it means 'movement in a straight line'. It doesn't. It means that definition, nothing more or less. A simple operator $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ (a translation operator, moving vectors in a straight line) is NOT LINEAR (if $\mathbf{b} \neq \mathbf{0}$). A transformation that rotates vectors around the origin (circular motion) WILL be a linear transformation.

One thing's that's good to check:
Property: ALL linear transformations $T : V \to W$ will map $\mathbf{0} \in V$ to $\mathbf{0} \in W$.

(Lame) Examples:
The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ where $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$ is the zero transformation. It's linear. Note that the image (output, whatever) is unique in the sense that each vector can have only one image, but multiple vectors can have the same image.
The identity transformation $\mathbf{1}_{\mathbb{R}^n}$ maps every vector $\mathbf{x} \in \mathbb{R}^n$ back to itself. It's linear...

Theorem: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a LINEAR transformation then T can be written as a matrix based transformation:

$$T(\mathbf{x}) = A\mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n$$

with A an $m \times n$ matrix of the form

$$A = \left[\ T(\mathbf{u}_1) \mid T(\mathbf{u}_2) \mid \cdots \mid T(\mathbf{u}_n)\ \right],$$

with $\mathbf{u}$ the unit vectors for $\mathbb{R}^n$.

Example: The projection onto a subspace of $\mathbb{R}^n$, $\text{Proj}_V(\mathbf{x})$, from $\mathbb{R}^n$ to $\mathbb{R}^n$, is a linear transformation.

Here's a simple linear transformation: $\text{Proj}_{\mathbf{y}}(\mathbf{x})$, $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The projection of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is simply $\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$, that for $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$ so the whole thing is $\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$. Lets look at its eigenvalues:

$$\left| \begin{array}{cc} 1/2-\lambda & -1/2 \\ -1/2 & 1/2-\lambda \end{array} \right| = \lambda^2 - \lambda + 1/4 - 1/4 = \lambda(\lambda - 1).$$

So, it has eigenvalues 0 and 1. This is actually a property of projections. Think about it. If A calculates a projection onto the subspace V then $A\mathbf{x}$ is INSIDE V, so $A(A\mathbf{x}) = A^2\mathbf{x}$. So, if we diagonalize, then

$$A^2 = PD^2P^{-1}, \qquad A = PDP^{-1},$$

so $D^2 = D$ and the eigenvalues are either 0 or 1. One could also observe that a projection preserves a certain number of directions and removes others, so 1 and 0.

Other examples include rotations. If we wanted to rotate points on $\mathbb{R}^2$ by 90° clockwise then we'd want to change $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (one point directly to the right) to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, (one point directly down). We'd also want to change $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (directly up) to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (directly to the right). As a result, the matrix is $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Lets look at those eigenvalues, so

$$\left| \begin{array}{cc} -\lambda & 1 \\ -1 & -\lambda \end{array} \right| = \lambda^2 + 1, \qquad \text{so} \qquad \lambda = \pm i.$$

This is a standard situation: complex eigenvalues relate to rotation, real eigenvalues relate to scaling. If you want to create an arbitrary rotation in $\mathbb{R}^2$ (and all rotations are 2 dimensional, really) then you get

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \qquad \theta = \text{degrees rotation clockwise.}$$

Example: Find a matrix A for the projection onto $V = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Answer: $A = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$. You can check, but this has two eigenvalues, a double of 1 and one of 0.

So, how do find the matrix to calculate the projection onto $V^\perp$? This is easy, actually.

$$\text{Proj}_{V^\perp}(\mathbf{x}) = \mathbf{x} - \text{Proj}_V(\mathbf{x}) = \mathbf{x} - A\mathbf{x} = I\mathbf{x} - A\mathbf{x} = (I - A)\mathbf{x}.$$

The operator $I - A$ will work as the projection matrix onto $V^\perp$ (when A is the matrix for the projection onto V).

Example: Given $T : \mathbb{R}^2 \to \mathbb{R}^3$ has $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, it's possible to calculate $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

Property: If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^l$ are linear transformations then $S \cdot T$ is a linear transformation on $\mathbb{R}^n \to \mathbb{R}^l$.

This is easy by creating matrix versions A for T, B for S. So $S \cdot T$ has BA as its matrix, and has to be a linear transformation.

Definitions For linear transformation $T : V \to W$:
The *Image* of T is the set $\{T(\mathbf{v}),\, \mathbf{v} \in V\}$.
The *Kernel* of T is the set $\{\mathbf{v} \,|\, T(\mathbf{v} = \mathbf{0} \in W\}$.

If we write $T : \mathbb{R}^n \to \mathbb{R}^m$ as A, $m \times n$, then what are these sets?

Exercises:

Can't find many on this subject...
Section 4.9: 4, 6.b), 10,

Pre-Exam Section:

Examples, Again

1. $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

2. Show that $U = \left\{ f \middle| f(1+x) = -f(1-x) \right\}$ is a subspace of $\mathcal{F}(0,2)$.

3. Solve the linear system $\begin{aligned} 3x_1 + x_2 + x_3 &= 5 \\ -x_1 + x_2 - 3x_3 &= -3 \\ 2x_1 + x_2 &= 3. \end{aligned}$

4. Solve the linear system $\begin{aligned} x_1 - x_2 - x_3 &= -1 \\ -2x_1 + 2x_2 + x_3 &= -1 \\ x_1 - x_2 &= 3. \end{aligned}$

5. Find a basis for $U = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $U^{\perp}$.

6. Is $A = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$ invertible? Now calculate the inverse.

7. Find the best fit for the points (x, y) : $(0, 2)$, $(1, -3)$, $(2, -2)$, for a line $a_0 + a_1 x = y$.

8. Find the eigenvalues and vectors to $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 2 & -2 \\ -1 & 0 & 3 \end{bmatrix}$.

9. Calculate $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$ for the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^2$, where

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad T\left(\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Answers:

1.

$$= \det \begin{bmatrix} u_1 & 1 & 2 \\ u_2 & 2 & -3 \\ u_3 & -1 & 1 \end{bmatrix} = u_1 \begin{vmatrix} 2 & -3 \\ -1 & 1 \end{vmatrix} - u_2 \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} + u_3 \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (-1) - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} (3) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (-7) = \begin{bmatrix} -1 \\ -3 \\ -7 \end{bmatrix}.$$

2. As usual, three things to check. First, the zero vector in $\mathcal{F}$ is the zero function $z(x) = 0$. It's in U since

$$z(1+x) = 0 = -z(1-x).$$

Next, we have to check multiplication, so $f \in U$ and $a \in \mathbb{R}$, and

$$(af)(1+x) = af(1+x) = a(-f(1-x)) = -af(1-x) = -(af)(1+x),$$

and af is in U. U is closed under scalar multiplication.

Finally, addition: $f, g \in U$ so

$$(f+g)(1+x) = f(1+x) + g(1+x) = -f(1-x) - g(1-x) = -(f+g)(1-x)$$

and $(f+g)$ is in U and so U is closed under vector addition.

3. Row reduction leads to

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

No contradiction, one free variable, $x_3 = t$. So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - x_3 \\ -1 + 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} t.$$

4. Row reduction leads to

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

(or something similar with non-zero values above that last pivot). This has $0 = 1$ as the last row, so a contradiction and no solution.

5. Arranging the vectors as rows of a matrix like so:

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 0 & 2 & -3 \\ 2 & 3 & 1 & 0 \end{bmatrix} \longrightarrow \text{Row Reduction} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The row space of A is equal to U, and has a basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$.

The null space of A is equal to $U^{\perp}$. We look to the solution set of $A\mathbf{x} = \mathbf{0}$, which is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 + 3x_4 \\ x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} s$$

using $t = x_3$ and $s = x_4$. So $U^{\perp}$ has $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ as a basis.

6. The inverse is $\begin{bmatrix} -2 & -1 & -4 \\ -2 & -1 & -3 \\ 1 & 1 & 2 \end{bmatrix}$.

7. The data matrix is $D = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, the first column relates to a_0, so it has to be in every row. The second column is the a_1 related row. Using Gram Schmidt on the columns of D, we get an orthogonal basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. The right hand side is $\mathbf{y} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$, with projection onto $\mathrm{Col}(D) = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$. Solving for a_0 and a_1 uses the ORIGINAL D matrix, so

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right].$$

The best fit is $1 - 2x = y$.

8. The eigenvalues are 4, 2 and 2 (double root for 2). The eigenvectors for 2 are $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. The eigenvector for 4 is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

9. First, we need to write $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ as a combination of the three vectors we know about.

So:

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \end{array}\right].$$

That reduces to $\left[\begin{array}{ccc|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$, so

$$\begin{aligned} T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) &= T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) - 3T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}. \end{aligned}$$